# The Geometry of Point and Line Reflections

**Notation:**

* Transformation *A* then *B* is written *BA.*
* Transformations *AA* may be written as *A2.*
* The identity (“do nothing”) is written *I.*
* Points are written as upper case, lines as lower case.
* Reflection in a point A is written as *A.* (Same as half turn about A).
* Reflection in line m is written *m*.

**Manipulating notation:**

As they are reflections, these transformations are self-inverse. *A2=I; a2 = I.*

A string of transformations that form a circuit (= *I*) may be rewritten, preserving cyclic order. For example: If *ABcDeF = I,* then *cDeFAB = I*.

**Three useful results:**

Complete the sentences with a description of the transformation and its magnitude in terms of the givens:

1. If **a** and **b** intersect, the product *ba* is equivalent to …..
2. If **a** and **b** are parallel, the product *ba* is equivalent to …..
3. The product of two point reflections *QP* is equivalent to …..

**Can you show that the following statements are equivalent?**

|  |  |
| --- | --- |
| Two lines **m** and **n** are perpendicular | *(mn)2 = I (mn ≠ I)* |
| The point **P** lies on the line **m** | *(aP)2 = I* |
| Three lines **a, b, c** are parallel | *(abc)2 = I* |
| The four points **PQRS** make a parallelogram | *PQRS = I* |
| **R** is the mid point of **PQ** | *PRQR = I (P ≠ Q)* |
| **m** is the perpendicular bisector of **PQ** | *PmQm = I* |
| Line **c** bisects one angle between lines **a** and **b**. | *acbc = I* |
| Point **R** divides the line joining **P** to **Q** in the ratio 2:1 | *PR(QR)2 = I* |
| Lines **a** and **b** cut at right angles at point **P** | *abP = 1* |
| **M** is the centroid of points **A, B, C** | *AMBMCM = I* |
| Line **h** is the altitude perpendicular to line **a** in the triangle formed by **a, b, c**. | *ahabchbc = I*  *(abc)2 ≠ I* |

G. Thomsen (1933); The Treatment of Elementary Geometry by a Group-Calculus; The Mathematical Gazette, Vol. 17, No. 225 (Oct., 1933), pp. 230-242

URL: http://www.jstor.org/stable/3607859 .Accessed: 06/06/2014 10:23

# Summations (S7)

Simple AP grid.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |

*By folding:* Fold in half vertically (first row 5, 11s; 5 x 13 etc). Fold in half horizontally (each pile = 40; so 25 x 40 = 1000)*. By turning:* Copy and turn one through 180°. Every cell = 20. 100 cells = 2000. Halve because 2 sheets = 1000)

## Addition square



Fold left to right and bottom to top. We are left with 25 piles of 121. 121x25=3025.  
(*or* rotate 90°, then fold about horizontal line)

## Multiplication table

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

**Solution**:

Add these by folding in half (right to left), then half again (bottom to top). Now you have a 5 x 5 square with each total = 121. Total = 121 x 25 = 3025. Prove this always works and generalise.

## AP by AP (general - not square)

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 8 | 15 | 22 | 29 | 36 | 43 | 50 | 57 | 64 |
| 4 | 11 | 18 | 25 | 32 | 39 | 46 | 53 | 60 | 67 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 10 | 17 | 24 | 31 | 38 | 45 | 52 | 59 | 66 | 73 |
| 13 | 20 | 27 | 34 | 41 | 48 | 55 | 62 | 69 | 76 |
| 16 | 23 | 30 | 37 | 44 | 51 | 58 | 65 | 72 | 79 |
| 19 | 26 | 33 | 40 | 47 | 54 | 61 | 68 | 75 | 82 |
| 22 | 29 | 36 | 43 | 50 | 57 | 64 | 71 | 78 | 85 |
| 25 | 32 | 39 | 46 | 53 | 60 | 67 | 74 | 81 | 88 |

**Solution:   
*By turning:*** Copy chart and turn through 180°. Sum numbers that cover one another then halve. (89 x number of cells)/2. Prove that this always works and generalise.

## GP by GP

## I can see what to do with products, but not sums!

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 |
| 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 |
| 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 |
| 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 | 16384 |
| 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 | 16384 | 32768 |
| 128 | 256 | 512 | 1024 | 2048 | 4096 | 8192 | 16384 | 32768 | 65536 |
| 256 | 512 | 1024 | 2048 | 4096 | 8192 | 16384 | 32768 | 65536 | 131072 |
| 512 | 1024 | 2048 | 4096 | 8192 | 16384 | 32768 | 65536 | 131072 | 262144 |

# Two-ness tasks

**Flipping cups**

You have N cups, all pointing upwards initially.

On any move, you can turn over any M of them.

Is it possible to have all N cups point downwards?

Quite obviously, if the problem is solvable for a given pair M, N then it is solvable for the pair qN, qM, with q a positive integer. For a long time I thought that the converse is also true, i.e. that the problems for (N, M) and the (reduced) pair M/***[gcd](http://www.cut-the-knot.org/blue/chinese.shtml#gcd)***(M, N), N/***[gcd](http://www.cut-the-knot.org/blue/chinese.shtml#gcd)***(M, N) are equivalent.

I was advised by the Zbarsky family that they are not. (For example, for N = 3 and M = 2 the problem has no solution. However, it is solvable for N = 6 and M = 4 in just three steps.)

For M and N are [***mutually prime***](http://www.cut-the-knot.org/arithmetic/Divisibility.shtml), the puzzle is solvable wherever M is odd, and unsolvable otherwise. Why?

It is easy to see that when N is odd and M is even, the puzzle is unsolvable. Indeed, assign number ±1 to each of the triangles depending whether it points up or down. Note the product Π of all the assigned numbers. When all triangles point upwards, Π = 1. When all point downwards, Π = -1. If M is even then flipping M triangles does not affect Π, and therefore, in this case, it is impossible to flip all the triangles.

If M = N - 1, there are C(N, N-1) = N [***combinations***](http://www.cut-the-knot.org/Generalization/cuttingcircle.shtml#comb) of N-1 elements out of N. Each of N elements enters N-1 of the combinations. Carrying out the flips corresponding to all N combinations, will turn each of the N triangles N-1 times, which is an odd number, and therefore leave it in a position different from the one it was originally in, i.e., upside down.

It remains to be shown that when gcd(M, N) = 1 and M odd the puzzle is always solvable. One solution can be found at [***my blog***](http://www.mathteacherctk.com/blog/2010/07/flipping-and-proving/).

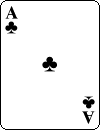
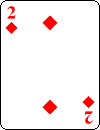
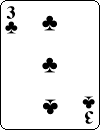
It so happens that the case of an odd N is much easier than the case where N is even. For odd N, the problem can always be solved in 3 steps. When N is even, one needs at least 4 moves.

# Dual Problems

“Given three points in the plane, ﬁnd a fourth point such that the sum of its distances to the three given points is a minimum.” (Fermat, 17th Century)

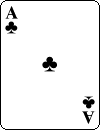
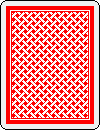
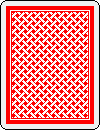
“Given any triangle, circumscribe the largest possible equilateral triangle about it.”  
Annales de Math´ematiques Pures et Appliqu´ees”, edited by [J. D. Gergonne](http://en.wikipedia.org/wiki/Joseph_Diaz_Gergonne), vol. I (1810-11) p. 384

**Hummer's One-Two-Three Trick**

**Effect:** Any Ace, 2 and 3 are placed face up in a row on the table. Turn away, and invite the victim to mentally choose one of the cards, turn it over on the table, and then also turn over the other two cards, *having first switched their positions*. You collect the cards, cut the packet repeatedly, and place them face down in a row on the table again. The victim guesses which card they mentally choose, and turns that one over *giving no hint as to whether they succeeded or not*. You instantly either congratulate them, or express disappointment at their choice and turn over the correct card, saying ``I think you meant this one.'' The trick may be repeated, and some audiences get more intrigued with each performance!

**Method:** Pick up the cards so that from top to bottom you have the rightmost card, the middle card, and the leftmost card. Cut cards to the bottom, one or two at a time, over and over with no obvious pattern, while secretly keeping count. Stop once 10, or 13, or 16, have been moved. Deal out the top card to the middle, the second card to the right, and the third to the left, and adjust your mental image of these three positions to read 3, 2, 1 from left to right (a reversal of the earlier placements). Invite the victim to turn over whichever card they think was theirs, say it's in position *i*. If the card with value *i* is in this position, congratulate them for an inspired guess, otherwise a card with value *j* not equal to *i* is in position *i*, and you quickly turn over the card in the third position *k* not equal to *i* or *j* and correct their choice.

For instance, if they turn over the Ace in position 3 (the left), you turn over the middle card (position 2), and that is indeed the card they first selected.

**Mathematics:** This trick is about permutations in the symmetric group on {1,2,3}, which we take to represent the initial positions of the Ace, 2 and 3. The victim's actions effect an unknown transposition . The manner in which you pick up the three cards reverses their order, which is equivalent to the permutation (1 3), another transposition. The cutting of s = 1 (mod 3) cards to the bottom of the packet boils down to the 3-cycle (1 3 2), and the final placement of the cards on the table (disregarding the left-right switch for now) is the same as the transposition (1 2). Since (1 2)(1 3 2)(1 3)  = , using the convention that permutations are multiplied from right to left, we see that knowing the identify of just one of the cards reveals what  is: the left-right switch is just to make it less obvious.

**Bonus points:** Different pick-up and dealing orders (and even numbers of cut cards) may be used to the same effect, provided that the algebra works out as above. If you can compute the permutation products on the fly, you can let the victim determine the pick-up order and number of cards cut, because an appropriate dealing order will ensure that the trick still works.

**Source:** This invention of Bob Hummer's, as adapted by Max Abrams, appears in Bob Longe's *Easy Magic Tricks* (Sterling, 1994), where Martin Gardner is also credited!